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ONDOKUZ MAYIS UNIVERSITY
INSTITUTE OF GRADUATE STUDIES
DEPARTMENT OF MATHEMATICS

DECOMPOSITION OF A THIRD-ORDER DISCRETE-TIME LINEAR TIME-VARYING SYSTEM INTO ITS FIRST AND SECOND-ORDER COMMUTATIVE PAIRS

Master Thesis

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ACCEPTANCE AND APPROVAL OF THESIS

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ABSTRACT

DECOMPOSITION OF A THIRD-ORDER DISCRETE-TIME LINEAR TIME-VARYING SYSTEM INTO ITS FIRST AND SECOND-ORDER COMMUTATIVE PAIRS

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In this thesis, necessary and sufficient conditions for the decomposition of any third order discrete-time linear time-varying system into its first and second-order commutative pairs are presented. The commutativity conditions expressed by Theorem 3.1 by accepting the initial conditions as zero at the beginning are also examined if the initial conditions are different from zero, and the results are presented with Theorem 3.2. The results, including the decomposition formulas are verified by the examples solved by using MATLAB Simulink tool. The importance of the thesis subject in terms of engineering application, which includes features such as realization, sensitivity, robustness and stability are emphasized.

Keywords: Commutativity, Decomposition, Discrete Time-Varying Systems.

ÖZET

ÜÇÜNCÜ MERTEBEDEN AYRIK ZAMANLI DOĞRUSAL ZAMANLA DEĞİŞEN SİSTEMİN BİRİNCİ VE İKİNCİ MERTEBEDEN DEĞİŞMELİ ÇİFTLERE AYRIŞTIRILMASI

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Yüksek Lisans, Ocak/2022
Danışman: Prof. Dr. Mehmet Emir KÖKSAL

Bu tezde, herhangi bir üçüncü mertebeden ayrık zamanlı doğrusal zamanla değişen sistemin birinci ve ikinci mertebeden komütatif eşlenliklerine ayrıştırılması için gerekli ve yeterli koşullar sunulmaktadır. Teorem 3.1 ile başlangıçta başlangıç koşulları sıfır kabul edilerek ifade edilen değişebilirlik koşulları, başlangıç koşullarının sıfırdan farklı olup olmadığı durumda da incelenmiş ve sonuçlar Teorem 3.2 ile sunulmuştur. Ayrıştırma formüllerini içeren sonuçlar, MATLAB Simulink aracı kullanılarak çözülen örneklerle doğrulanmıştır. Tez konusunun gerçekleştirme, hassasiyet, sağlamlık ve kararlılık gibi özellikleri içeren mühendislik uygulamaları açısından önemi vurgulanmıştır.

Anahtar Sözcükler: Değişmelilik, Ayrışma, Ayrık Zamanla Değişen Sistem.

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I cannot forget my family members for their many years of support, guidance. endless encouragement and prayers, more especially my Mother Kadija Ahmed Hamud, my father Hassan Abdullahi, my brother Daud Mohamed Salad.

Mohamed Hassan Abdullahi

CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

The realization of many engineering systems is done by cascading simpler systems. This is very important in the design of digital systems (Nguyen et al, 2015; Arnol et al, 2017; Lyons, 2011). Although the connection order of these subsystems depends on the specific design and engineering experience used, changing the connection order of the subsystems without changing the main function of the total system (commutativity) can have positive results when sensitivity, stability, noise exposure and durability (system performance) are taken into account. (Lyons, 2011; Koksal and Koksal ME, 2015). It is expected to contribute to the development studies in the design of systems and devices in which sequential circuits are used in communication systems, which are especially important in engineering matters today. For this reason, commutativity is very important in terms of practical applications in electrical and electronic engineering.

The first publication on commutativity was reviewed in 1977 for first-order time-varying analog systems with zero initial conditions (Marshall, 1977), and the results here were generalized to higher-order analog systems with non-zero initial conditions in the 1980s (Koksal, 1982; Koksal , 1988). Since all the studies on commutativity (Koksal and Koksal ME, 2013, 2015; Koksal, ME, 2019) are related to the commutativity of analog systems and today's technology is based on digital systems rather than analog systems (Nguven et al, 2013; Phat et al, 2011; Parlakçı et al, 2014; Shen et al, 2013; Tabataeipour, SM, 2013; Yao et al, 2014; Zhang et al, 2014; Zhong et al, 2013) there is a serious gap that needs to be studied on the commutativity of digital systems. Although the general conditions for the commutativity of time-varying linear systems have been determined, the properties of the difference equations that describe most physical phenomena in the literature have not been examined in terms of commutativity and there are no concrete results about them except a few articles.

In engineering, most systems consist of interconnected subsystems (cascade or chain link), with one output being the other's input. This connection is used very frequently, especially in electrical circuits. Therefore, each of the results to be found within the scope of the research on how the connection or process sequence should be in order for such systems to be less sensitive to noise and similar external effects will result in the original value of the subject. As a concrete example, in the design of a double-stage electrical filter that provides a given input-output characteristic (these filters may be of different types), it is important for engineering that which order will be more advantageous in terms of noise, signal/noise ratio or sensitivity compared to element values. Theoretically, the existence of one subsystem before or after the other is functionally insignificant (which will be ensured by the validity of the commutativity conditions), but it is important in practice because of distortions. Concrete examples based on engineering or industry will increase the practical importance of the study in addition to the theoretical findings and will provide the opportunity to evaluate the contributions made technologically.

On the other hand, fixed parameter systems are obtained by making some assumptions and idealizations in real physical systems. An element value that is accepted as constant, albeit a little, may have a time-varying parameter. Changing this over time will cause noise and similar distortions or deviations on the system, unlike the ideal situation. In cascading systems, for example, it will be tried to solve how the distortions occurring in the first stage can be prevented by using time dependent elements deliberately in the second stage, by using the previously established theory about commutativity. This approach will contribute to the emergence of new theories and the development of the existing theory. The application of commutativity conditions and theorems to discrete time engineering systems has not been studied so far, and this is a serious shortcoming.

The subject of commutativity is handled in two ways in the study of physical systems/events. For example, if a physical system can be decomposed into two separate subsystems (physically or by model) in accordance with the definition of commutativity, the rank change feature of these subsystems is defined as the commutativity of the subsystems. It is important for engineering to examine the changes in the properties (sensitivity, stability, durability, etc.) that are important in practice, while keeping the input-output relationship of a physical system the same, and to use the appropriate order. In another way, if the physical system is a subsystem of a larger system, examining the changes in the performance of the main system in the event that this subsystem is changed in order of interaction with other components that follow; in which cases these changes will not impair the actual performance of the system.

The subject of the thesis is to investigate the commutative conjugate subsystems with an existing physical system (subsystem) and to examine the sequence change feature of the subsystems that will be formed together when they are connected sequentially, and the features that are important in terms of engineering (sensitivity, durability, sensitivity, etc.). It is true that the mathematical model is derived from the physical system, but finding (designing) the physical system that implements a mathematical model is a more important and difficult task in terms of engineering. In different connection structures, the original system will be considered as cascade connection of subsystems that are commutative conjugates of this system. The relationship between subsystems has been mentioned mathematically with general theorems as stated in the previous reference (Koksal, & Koksal, 2011).

The following example can be considered in regard of the applications that are specific to physical systems.

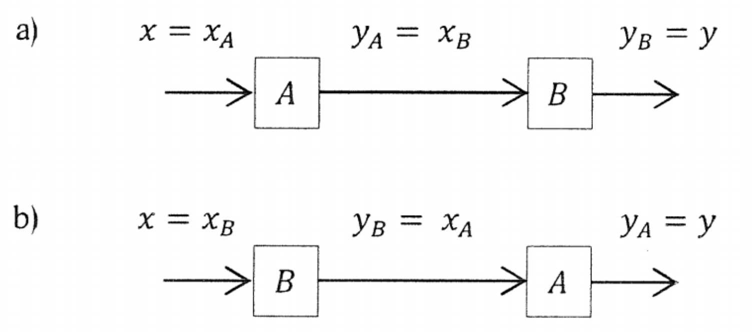


Figure 1.1. Subsystems A and B are connected in cascade way

If part A of the system shown in Figure 1.1 represents the transmitter of a communication system and part B represents the receiver, if A and B are commutative, then AB and BA will perform the same function (function, input-output correlation). However, different signals will be emitted on the line between A and B, which represents the transmission medium. Broadcasting the signal sometimes from A and sometimes B transmitter will cause differentiation of the signal in the propagation environment, but there will be no signal loss or alienation on the receiver side due to commutativity feature. The propagation of different signals in different time periods in the transmission medium, as well as the commutative conjugation of A and B systems and the knowledge of the designer, will contribute to the concealment of the information desired to be transmitted. (Commutative conjugates of a time-varying system A, that is, system B, are not common for digital systems, and there is no theorem regarding commutativity conditions in the literature.) This contribution of commutativity in cryptology can be the subject of further scientific studies when combined with the application of chaotic systems in cryptology (Willigenburg et al., 2015; Li et al, 2016).

It is known that designs made by cascading connections in electrical circuits (filters) are much more advantageous in terms of stability, durability and sensitivity than others. The realization of a time-varying system as a cascade connection of two commutative subsystems is important for the designs of systems with similar properties. As a matter of fact, examples of this have been observed theoretically in the simulations. In addition, since the commutative equivalents of a time-varying system are very few and special systems (theoretically, their discovery has not been sufficiently studied and is not yet widely known in the scientific world), using them on separate sides of a communication line is an approach similar to that of chaos in cryptology. may also enable its use.

If the physical or model implementation of a system can be in the form of a sequential connection of two subsystems as in Figure.1.1a or 1.1b as a cause-and-effect relationship, the conditions under which this separation can be performed on subsystems A and B will be examined in light of the general conditions of the commutativity. Since one of these distinctions may be superior to the other, it is important for implementation. At the end of these researches, results will be expected that will be important for engineering applications for optimal system design and control.

Commutativity can be applied to any field (electrical, mechanical, hydraulic, socioeconomic, biological, etc.) modeled with time-varying parameters. Giving specific (specific) application areas and examples will be researched and made within the scope of the thesis. Since the subject is new and has just begun to be studied, a sufficient number of concrete applications have not been proposed yet. However, in some cases, conceptual and mathematical issues precede practical applications, and applications and physical examples come later. For example, the memristor element was found (discovered) long after its definition by Chua, and fuzzy logic applications were widely used in the field (especially in the control field) after Fetwies introduced the idea of fuzzy logic as an alternative to "discrete" logic. Or the use of Chaos in cryptology long after the mathematical foundations of Chaos theory have been studied in detail.

The first time to conduct a study on decomposition of time varying linear systems was in 2016. The decomposition of second-order linear systems that varies with time and their conditions had been proved into stage connected first-order abelian pairs (Koksal, 2016). After that, followed by the study on the decomposition of a third-order linear systems with continuous time varying into sequentially connected second and first-order abelian pairs which was again examined alongside with the sub-system's clear results by (Koksal, Yakar, 2019).

Lastly, the decomposition of the fourth-order linear continuous-time time varying system into sequentially or two-stage connected second-order abelian pairs, into third and first-order abelian pairs was presented (Ibrahim, 2019).

2. COMMUTATIVITY

Let the subsystems A and B in Fig. 1.1 be defined by the following linear discrete-time time-varying difference equations:

$$A: a_1(k)y_A(k+1) + a_0(k)y_A(k) = x_A(k), k \geq 0 \quad \textbf{(2.1a)}$$

$$y_A(0) = y_{A0}, \quad \textbf{(2.2a)}$$

$$B: b_2(k)y_B(k+2) + b_1(k)y_B(k+1) + b_0(k)y_B(k) = x_B(k), k \geq 0 \quad \textbf{(2.1b)}$$

$$y_B(0) = y_{B0}, y_B(1) = y_{B1}, \quad \textbf{(2.2b)}$$

Note that $a_1(k)$ and $b_2(k)$ must not be zero for all $k \geq 0$ for the solvability of A and B. From the connection AB shown in Fig. 1.1a, it follows that

$$x(k) = x_A(k), \qquad y_A(k) = x_B(k), \qquad y_B(k) = y(k), k \geq 0 \quad \textbf{(2.3)}$$

With this and Eqs. (2.1a) and (2.2a), the difference equation between $x(k)$ and $\mathrm{y}(k)$ can be obtained as follows: Write Eq. (2.2a) for $k+1$

$$b_2(k+1)y_B(k+3) + b_1(k+1)y_B(k+2)$$
$$+b_0(k+1)y_B(k+1) = x_B(k+1), k \geq 1$$

Insert $y_B(k) = y(k)$, $x_B(k) = y_A(k)$, from Eq. (2.3)

$$b_2(k+1)y(k+3) + b_1(k+1)y(k+2) + b_0(k+1)y(k+1) = y_A(k+1), k \geq 1$$

and eliminate $y_A(k+1)$ using Eq. (2.1a), (2.3),

$$b_2(k+1)y(k+3) + b_1(k+1)y(k+2) + b_0(k+1)y(k+1)$$
$$= \frac{x_A(k) - a_0(k)y_A(k)}{a_1(k)} = \frac{x(k) - a_0(k)y_A(k)}{a_1(k)} = \frac{x(k) - a_0(k)x_B(k)}{a_1(k)}, k \geq 1.$$

Finally using Eqs. (2.2a) and (2.3)

$$b_2(k+1)y(k+3)+b_1(k+1)y(k+2)+b_0(k+1)y(k+1)$$

$$=\frac{x(k)-a_0(k)[b_2(k)y_B(k+2)+b_1(k)y_B(k+1)+b_0(k)y_B(k)]}{a_1(k)}$$

$$=\frac{x(k)-a_0(k)[b_2(k)y(k+2)+b_1(k)y(k+1)+b_0(k)y(k)]}{a_1(k)}, k\geq 1$$

Multiplying by $a_1(k)$ and rearranging we obtain:

$$AB{:}\, a_1(k)b_2(k+1\,)y(k+3)$$
$$+[a_1(k)b_1(k+1\,)+a_0(k)b_2(k)]y(k+2)$$
$$+[a_1(k)b_0(k+1)+a_0(k)b_1(k)]y(k+1)$$
$$+a_0(k)b_0(k)y(k)=x(k) \quad \textbf{(2.4a)}$$

With Eq. (2.3), the initial conditions in Eqs. (2.1b, 2.2b), and Eq. (2.2a) for $k=0$, the initial conditions for $y(k)$ are obtained as

$$y(0)=y_B(0)=y_{B0},$$
$$y(1)=y_B(1)=y_{B1},$$
$$y(2)\;=y_B(2)=\frac{x_B(0)-b_1(0)y_B(1)-b_0(0)y_B(0)}{b_2(0)}$$
$$=\frac{y_{A0}-b_1(0)y_{B1}-b_0(0)y_{B0}}{b_2(0)} \quad \textbf{(2.4b)}$$

From the connection BA shown in Fig. 1.1b, it follows that

$$x(k)=x_B(k),\qquad y_B(k)=x_A(k),\qquad y_A(k)=y(k), k\geq 0 \quad \textbf{(2.5a)}$$

With this and Eqs. (2.1a) and (2.2a), the difference equation between $x(k)$ and y(k) can be obtained as follows: Write Eq. (2.1a) for $k+2$

$$a_1(k+2)y_A(k+3)+a_0(k+2)y_A(k+2)=x_A(k+2), k\geq 2$$

Insert $y_A(k)=y(k), x_A(k)=y_B(k),$ from Eq. (2.5)

$$a_1(k+2)y(k+3)+a_0(k+2)y(k+2)=y_B(k+2), k\geq 2$$

Eliminate $y_B(k+2)$ using Eq. (2.2a), (2.5)

$$a_1(k+2)y(k+3)+a_0(k+2)y(k+2)$$
$$=\frac{x_B(k)-b_1(k)y_B(k+1)-b_0(k)y_B(k)}{b_2(k)}$$
$$=\frac{x(k)-b_1(k)x_A(k+1)-b_0(k)x_A(k)}{b_2(k)},k\geq 2$$

And from Eqs. (2.1a) and (2.5), eliminate $x_A(k), x_A(k+1)$;

$$a_1(k+2)y(k+3)+a_0(k+2)y(k+2)=\frac{x(k)}{b_2(k)}$$
$$-\frac{b_1(k)[a_1(k+1)y(k+2)+a_0(k+1)y(k+1)]}{b_2(k)}$$
$$-\frac{b_0(k)[a_1(k)y(k+1)+a_0(k)y(k)]}{b_2(k)}$$

Finally, by multiplying with $b_2(k)$ and rearranging we obtain

$$BA: b_2(k)a_1(k+2)y(k+3)$$
$$+[b_2(k\ )a_0(k+2)+b_1(k)a_1(k+1)]y(k+2)$$
$$+[b_1(k\ )a_0(k+1)+b_0(k)a_1(k)]y(k+1)$$
$$+b_0(k)a_0(k)y(k)=x(k) \quad \textbf{(2.6a)}$$

With Eqs. (2.5), (2.1a) the initial conditions for $y(k)$ are obtained as;

$$y(0)=y_A(0)=y_{A0},$$
$$y(2)\ =y_A(2)=\frac{x_A(1)-a_0(1)y_A(1)}{a_1(1)}=\frac{y_B(1)-a_0(1)y(1)}{a_1(1)}$$

$$=\frac{y_{B1}-a_0(1)\dfrac{y_{B0}-a_0(0)y_{A0}}{a_1(0)}}{a_1(1)}$$

$$= \frac{a_1(0)y_{B1} - a_0(1)y_{B0} + a_0(1)a_0(0)y_{A0}}{a_1(0)a_1(1)} \quad \textbf{(2.6b)}$$

On the other hand, the connection AB and BA of sub-systems A and B are requested to be equivalent to the same system (say C) due to the commutativity. Let C be defined by

$$c_3(k)y(k+3) + c_2(k)y(k+2) + c_1(k)y(k+1)$$
$$+c_0(k)y(k) = x(k), k \geq 0, \quad \textbf{(2.7a)}$$

$$y(2), y(1), y(0) \quad \textbf{(2.7b)}$$

If Eqs. (2.4a) and (2.6a) defining AB and BA are compared with Eq. (2.7a) defining C, we have

$$c_3(k) = a_1(k)b_2(k+1) = b_2(k)a_1(k+2) \quad \textbf{(2.8a)}$$
$$c_2(k) = a_1(k)b_1(k+1) + a_0(k)b_2(k)$$
$$= b_2(k)a_0(k+2) + b_1(k)a_1(k+1), \quad \textbf{(2.8b)}$$
$$c_1(k) = a_1(k)b_0(k+1) + a_0(k)b_1(k)$$
$$= b_1(k)a_0(k+1) + b_0(k)a_1(k), \quad \textbf{(2.8c)}$$
$$c_0(k) = a_0(k)b_0(k) = b_0(k)a_0(k). \quad \textbf{(2.8d)}$$

In addition, if the equations (2.4b) and (2.6b) are compared;

$$y(0) = y_{B0} = y_{A0},$$
$$y(1) = y_{B1} = \frac{y_{B0} - a_0(0)y_{A0}}{a_1(0)},$$
$$y(2) = \frac{y_{A0} - b_1(0)y_{B1} - b_0(0)y_{B0}}{b_2(0)} = \frac{a_1(0)y_{B1} - a_0(1)y_{B0} + a_0(1)a_0(0)y_{A0}}{a_1(1)a_1(0)}.$$

Using the first and the second equations, in the second and third equations, we write the result as

$$y(0) = y_{B0} = y_{A0} \quad \textbf{(2.9a)}$$

The last equation implies that if $y_{A0} \neq 0$, it must be true that

$$\frac{a_1(0) - b_1(0) + b_1(0)a_0(0) - a_1(0)b_0(0)}{b_2(0)} = \frac{1 - a_0(0) - a_0(1) + a_0(1)a_0(0)}{a_1(1)} \quad \textbf{(2.10)}$$

3. DERIVATION OF DECOMPOSITION FORMULAS AND THEOREMS

Decomposition is very significant method that is used in many various systems to solve the problems that arise in engineering fields and improve the stability of the system. Decomposition divides a high-order linear systems into lower-order pairs (commutative pairs in the scope of this thesis). Decomposition simplifies the break-down of complex problems into simpler and solvable parts. These reduced parts can then be easily analyzed or resolved separately as they function easily. When the problem is not reduced, it is so difficult to solve in many cases. Working and analyzing higher level systems as a whole is much more difficult than dividing it into subdivisions and solving each part at a time. To explore each subsystem in a more detail, the systems should be broken down into subdivisions or parts. Decomposition provides the ability to understand how a complex system functions and makes it easy to comprehend when using decomposition.

From equation (2.8a)

$$b_2(k+1) = \frac{b_2(k)a_1(k+2)}{a_1(k)};$$

$k = 0$

$$b_2(1) = \frac{b_2(0)a_1(2)}{a_1(0)} = \frac{b_2(0)}{a_1(0)}a_1(2),$$

$k = 1$

$$b_2(2) = \frac{b_2(1)a_1(3)}{a_1(1)} = \frac{a_1(3)}{a_1(1)}b_2(1) = \frac{b_2(0)a_1(2)a_1(3)}{a_1(0)a_1(1)},$$

$k = 2$

$$b_2(3) = \frac{a_1(4)}{a_1(2)}b_2(2) = \frac{b_2(0)a_1(2)a_1(3)a_1(4)}{a_1(0)a_1(1)a_1(2)} = \frac{b_2(0)a_1(3)a_1(4)}{a_1(0)a_1(1)},$$

$k = 3$

$$b_2(4) = \frac{b_2(3)a_1(5)}{a_1(3)} = \frac{a_1(5)}{a_1(3)}b_2(3) = \frac{b_2(0)a_1(3)a_1(4)a_1(5)}{a_1(0)a_1(1)a_1(3)} = \frac{b_2(0)a_1(4)a_1(5)}{a_1(0)a_1(1)},$$

$k = 4$

$$b_2(5) = \frac{b_2(4)a_1(6)}{a_1(4)} = \frac{a_1(6)}{a_1(4)}b_2(4) = \frac{b_2(0)a_1(4)a_1(5)a_1(6)}{a_1(0)a_1(1)a_1(4)} = \frac{b_2(0)a_1(5)a_1(6)}{a_1(0)a_1(1)},$$

$\vdots$

Generalizing the last equation,

$$b_2(k) = \frac{b_2(0)a_1(k)}{a_1(0)a_1(1)}a_1(k+1) \tag{3.1a}$$

Again, by using Eq. (2.8a) and replacing $b_2(k)$ in Eq. (2.8a) with its equivalence in Eq. (3.1a)

$$a_1(k+2) = \frac{c_3(k)a_1(0)a_1(1)}{b_2(0)a_1(k)a_1(k+1)};$$

$k = 0,$

$$a_1(2) = \frac{c_3(0)a_1(0)a_1(1)}{b_2(0)a_1(0)a_1(1)} = \frac{c_3(0)}{b_2(0)},$$

$k = 1$

$$a_1(3) = \frac{c_3(1)a_1(0)a_1(1)}{b_2(0)a_1(1)a_1(2)} = \frac{c_3(1)a_1(0)}{b_2(0)a_1(2)} = \frac{a_1(0)c_3(1)b_2(0)}{b_2(0)c_3(0)} = \frac{a_1(0)c_3(1)}{c_3(0)},$$

$k = 2$

$$a_1(4) = \frac{c_3(2)a_1(0)a_1(1)}{b_2(0)a_1(2)a_1(3)} = \frac{c_3(2)a_1(1)}{c_3(1)},$$

$k = 3$

$$a_1(5) = \frac{c_3(3)a_1(0)a_1(1)}{b_2(0)a_1(3)a_1(4)} = \frac{c_3(0)c_3(3)}{b_2(0)c_3(2)},$$

$k = 4$

$$a_1(6) = \frac{c_3(4)a_1(0)a_1(1)}{b_2(0)a_1(4)a_1(5)} = \frac{a_1(0)c_3(4)c_3(1)}{c_3(0)c_3(3)},$$

$k = 5$

$$a_1(7) = \frac{c_3(5)a_1(0)a_1(1)}{b_2(0)a_1(5)a_1(6)}$$
$$= \frac{c_3(5)a_1(0)a_1(1)b_2(0)c_3(2)c_3(0)c_3(3)}{b_2(0)c_3(0)c_3(3)a_1(0)c_3(4)c_3(1)} = \frac{a_1(1)c_3(5)c_3(2)}{c_3(1)c_3(4)},$$

$k = 6$

$$a_1(8) = \frac{c_3(6)a_1(0)a_1(1)}{b_2(0)a_1(6)a_1(7)} = \frac{c_3(0)c_3(6)c_3(3)}{b_2(0)c_3(5)c_3(2)},$$

$k = 7$

$$a_1(9) = \frac{c_3(7)a_1(0)a_1(1)}{b_2(0)a_1(7)a_1(8)} = \frac{a_1(0)c_3(7)c_3(4)c_3(1)}{c_3(0)c_3(6)c_3(3)},$$

$k = 8$

$$a_1(10) = \frac{c_3(8)a_1(0)a_1(1)}{b_2(0)a_1(8)a_1(9)} = \frac{a_1(1)c_3(8)c_3(5)c_3(2)}{c_3(1)c_3(7)c_3(4)},$$

$k = 9$

$$a_1(11) = \frac{c_3(9)a_1(0)a_1(1)}{b_2(0)a_1(9)a_1(10)} = \frac{c_3(0)c_3(9)c_3(6)c_3(3)}{b_2(0)c_3(8)c_3(5)c_3(2)},$$

⋮

After generalizing as $a_1(k)$, the following equation is obtained.

$$a_1(k) = \begin{cases} \frac{c_3(0)}{b_2(0)}, & k = 2 \\ a_1(0)\prod_{i=1}^{\frac{k}{3}} \frac{c_3(3i-2)}{c_3(3i-3)}, & k = 3,6,9,\ldots \\ a_1(1)\prod_{i=1}^{\frac{k-1}{3}} \frac{c_3(3i-1)}{c_3(3i-2)}, & k = 4,7,10,\ldots \\ \frac{c_3(0)}{b_2(0)}\prod_{i=1}^{\frac{k-2}{3}} \frac{c_3(3i)}{c_3(3i-1)}, & k = 5,8,11,\ldots \end{cases} \tag{3.1b}$$

To find $b_1(k)$, solving for $b_1(k+1)$ in Eq. (2.8b) and replacing $b_2(k)$ with Eq. (3.1a);

$$b_1(k+1) = \frac{b_2(0)a_1(k+1)}{a_1(0)a_1(1)}[a_0(k+2) - a_0(k)] + \frac{a_1(k+1)}{a_1(k)}b_1(k),$$

$k = 0,$

$$b_1(1) = \frac{b_2(0)}{a_1(0)}[a_0(2) - a_0(0)] + \frac{b_1(0)}{a_1(0)}a_1(1),$$

$k = 1,$

$$b_1(2) = \frac{b_2(0)a_1(2)}{a_1(0)a_1(1)}[a_0(3) - a_0(1)] + \frac{a_1(2)}{a_1(1)}b_1(1)$$

$$= \frac{b_2(0)a_1(2)}{a_1(0)a_1(1)}[a_0(3) - a_0(2)] + \frac{a_1(2)}{a_1(1)}\frac{b_2(0)}{a_1(0)}[a_0(2) - a_0(0)] + \frac{b_1(0)}{a_1(0)}a_1(2)$$

$$= \frac{b_2(0)a_1(2)}{a_1(0)a_1(1)}[a_0(3) - a_0(1) + a_0(2) - a_0(0)] + \frac{b_1(0)}{a_1(0)}a_1(2),$$

General form,

$$b_1(k) = \frac{b_2(0)a_1(k)}{a_1(0)a_1(1)}[a_0(k+1) - a_0(1) + a_0(k) - a_0(0)] + \frac{b_1(0)}{a_1(0)}a_1(k) \qquad \textbf{(3.1c)}$$

To find $b_0(k)$, we solve $b_0(k+1)$ in Eq. (2.8c) and replace $b_1(k)$ with Eq. (3.1c), then we arrive at the following equation.

$$b_0(k+1) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(k+1) - a_0(1) + a_0(k) - a_0(0)][a_0(k+1) - a_0(k)] + \frac{b_1(0)}{a_1(0)}[a_0(k+1) - a_0(k)] + b_0(k);$$

k=0,

$$b_0(1) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(1) - a_0(1) + a_0(0) - a_0(0)][a_0(1) - a_0(0)] + \frac{b_1(0)}{a_1(0)}[a_0(1) - a_0(0)] + b_0(0) = \frac{b_1(0)}{a_1(0)}[a_0(1) - a_0(0)] + b_0(0),$$

k=1,

$$b_0(2) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(2) - a_0(1) + a_0(1) - a_0(0)][a_0(2) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(2) - a_0(1)] + b_0(1)$$

$$= \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(2) - a_0(0)][a_0(2) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(2) - a_0(1)] + b_0(1)$$

$$= \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(2) - a_0(0)][a_0(2) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(2) - a_0(0)] + b_0(0).$$

For general values of k,

$$b_0(k) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(k) - a_0(0)][a_0(k) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(k) - a_0(0)] + b_0(0). \qquad \textbf{(3.1d)}$$

To derive $a_0(k)$, solving Eq. (2.8b), for $a_0(k+2)$ and substituting $b_2(k)$ and $b_1(k)$ in Eqs. (3.1a) and (3.1c),

$$a_0(k+2) = \frac{c_2(k)a_1(0)a_1(1)}{b_2(0)a_1(k)a_1(k+1)} - a_0(k+1) + a_0(1) - a_0(k) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1);$$

k=0

$$a_0(2) = \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)}a_1(1),$$

k=1,

$$a_0(3) = \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} - a_0(2) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1) = \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} - \frac{c_2(0)}{b_2(0)} + a_0(0),$$

k=2,

$$a_0(4) = \frac{c_2(2)a_1(0)a_1(1)}{b_2(0)a_1(2)a_1(3)} - a_0(3) - a_0(2) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1) = \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)} - \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} + a_0(1),$$

k=3,

$$a_0(5) = \frac{c_2(3)a_1(0)a_1(1)}{b_2(0)a_1(3)a_1(4)} - a_0(4) - a_0(3) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1) = \frac{c_2(3)a_1(0)a_1(1)}{b_2(0)a_1(3)a_1(4)} - \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)} + \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)}a_1(1),$$

k=4,

$$a_0(6) = \frac{c_2(4)a_1(0)a_1(1)}{b_2(0)a_1(4)a_1(5)} - a_0(5) - a_0(4) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1) = \frac{a_1(0)c_2(4)a_1(1)}{b_2(0)a_1(4)a_1(5)} - \frac{a_1(0)c_2(3)a_1(1)}{b_2(0)a_1(3)a_1(4)} + \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} - \frac{c_2(0)}{b_2(0)} + a_0(0),$$

k=5,

$$a_0(7) = \frac{c_2(5)a_1(0)a_1(1)}{b_2(0)a_1(5)a_1(6)} - a_0(6) - a_0(5) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1)$$

$$= \frac{a_1(0)c_2(5)a_1(1)}{b_2(0)a_1(5)a_1(6)} - \frac{a_1(0)c_2(4)a_1(1)}{b_2(0)a_1(4)a_1(5)} - \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} + \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)} + a_0(1),$$

k=6,

$$a_0(8) = \frac{c_2(6)a_1(0)a_1(1)}{b_2(0)a_1(6)a_1(7)} - a_0(7) - a_0(6) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1)$$

$$= \frac{a_1(0)c_2(6)a_1(1)}{b_2(0)a_1(6)a_1(7)} - \frac{a_1(0)c_2(5)a_1(1)}{b_2(0)a_1(5)a_1(6)} + \frac{a_1(0)c_2(3)a_1(1)}{b_2(0)a_1(3)a_1(4)}$$

$$- \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)} + \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)}a_1(1),$$

k=7,

$$a_0(9) = \frac{c_2(7)a_1(0)a_1(1)}{b_2(0)a_1(7)a_1(8)} - a_0(8) - a_0(7) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1)$$

$$a_0(9) = \frac{a_1(0)c_2(7)a_1(1)}{b_2(0)a_1(7)a_1(8)} - \frac{a_1(0)c_2(6)a_1(1)}{b_2(0)a_1(6)a_1(7)} - \frac{a_1(0)c_2(4)a_1(1)}{b_2(0)a_1(4)a_1(5)}$$

$$- \frac{a_1(0)c_2(3)a_1(1)}{b_2(0)a_1(3)a_1(4)} + \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} - \frac{c_2(0)}{b_2(0)} + a_0(0),$$

k=8,

$$a_0(10) = \frac{c_2(8)a_1(0)a_1(1)}{b_2(0)a_1(8)a_1(9)} - a_0(9) - a_0(8) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1)$$

$$= \frac{a_1(0)c_2(8)a_1(1)}{b_2(0)a_1(8)a_1(9)} - \frac{a_1(0)c_2(7)a_1(1)}{b_2(0)a_1(7)a_1(8)} + \frac{a_1(0)c_2(4)a_1(1)}{b_2(0)a_1(4)a_1(5)}$$

$$- \frac{a_1(0)c_2(1)}{b_2(0)a_1(2)} + \frac{a_1(0)c_2(5)a_1(1)}{b_2(0)a_1(5)a_1(6)} + \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)} + a_0(1),$$

k=9,

$$a_0(11) = \frac{c_2(9)a_1(0)a_1(1)}{b_2(0)a_1(9)a_1(10)} - a_0(10) - a_0(9) + a_0(1) + a_0(0) - \frac{b_1(0)}{b_2(0)}a_1(1)$$

$$= \frac{a_1(0)c_2(9)a_1(1)}{b_2(0)a_1(9)a_1(10)} - \frac{a_1(0)c_2(8)a_1(1)}{b_2(0)a_1(8)a_1(9)} + \frac{a_1(0)c_2(6)a_1(1)}{b_2(0)a_1(6)a_1(7)} - \frac{a_1(0)c_2(5)a_1(1)}{b_2(0)a_1(5)a_1(6)}$$

$$+ \frac{a_1(0)c_2(3)a_1(1)}{b_2(0)a_1(3)a_1(4)} - \frac{a_1(0)c_2(2)a_1(1)}{b_2(0)a_1(2)a_1(3)}$$

$$+ \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)}a_1(1).$$

Generalizing, we obtain

$$a_0(k) = \begin{cases} \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1), \quad k = 2 \\ \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=0}^{\frac{k-3}{3}} \dfrac{c_2(3i+1)a_1(1)}{a_1(3i+1)a_1(3i+2)} - \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=0}^{\frac{k-3}{3}} \dfrac{c_2(3i)a_1(1)}{a_1(3i)a_1(3i+1)} \\ \quad + a_0(0), \quad k = 3,6,9,\cdots \\ \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=1}^{\frac{k-1}{3}} \dfrac{c_2(3i-1)a_1(1)}{a_1(3i-1)a_1(3i)} - \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=1}^{\frac{k-1}{3}} \dfrac{c_2(3i-2)a_1(1)}{a_1(3i-2)a_1(3i-1)} \\ \quad + a_0(1), \quad k = 4,7,10,\cdots \\ \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=0}^{\frac{k-2}{3}} \dfrac{c_2(3i)a_1(1)}{a_1(3i)a_1(3i+1)} - \dfrac{a_1(0)}{b_2(0)} \displaystyle\sum_{i=1}^{\frac{k-2}{3}} \dfrac{c_2(3i-1)a_1(1)}{a_1(3i-1)a_1(3i)} \\ \quad - \dfrac{b_1(0)}{b_2(0)} a_1(1), \quad k = 5,8,11,\cdots \end{cases}$$

Finally, $a_0(k)$ in simplified form is

$$a_0(k) = \begin{cases} \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1), \quad k = 2 \\ \dfrac{a_1(0)a_1(1)}{b_2(0)} \displaystyle\sum_{i=0}^{\frac{k-3}{3}} \left[\dfrac{c_2(3i+1)}{a_1(3i+1)a_1(3i+2)} - \dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)} \right] \\ \quad + a_0(0), \quad k = 3,6,9,\cdots \\ \dfrac{a_1(0)a_1(1)}{b_2(0)} \displaystyle\sum_{i=1}^{\frac{k-1}{3}} \left[\dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)} - \dfrac{c_2(3i-2)}{a_1(3i-2)a_1(3i-1)} \right] \\ \quad + a_0(1), \quad k = 4,7,10,\cdots \\ \dfrac{a_1(0)a_1(1)}{b_2(0)} \displaystyle\sum_{i=1}^{\frac{k-2}{3}} \left[\dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)} - \dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)} \right] \\ \quad - \dfrac{b_1(0)}{b_2(0)} a_1(1) + \dfrac{c_2(0)}{b_2(0)}, \quad k = 5,8,11,\cdots \end{cases} \qquad \textbf{(3.1e)}$$

If the values of $a_1(k)$ in equation (3.1b) are put into the equation (3.1d), and arranged, the solution for $a_0(k)$ is found as in the following equation (3.2a);

$$a_0(k) = \begin{cases} -\frac{b_1(0)}{b_2(0)} a_1(1) + A(0), & k = 2 \\ \frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(0), & k = 3,6,9,\cdots \\ \frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(1), & k = 4,7,10,\cdots \\ \frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} - \frac{b_1(0)}{b_2(0)} a_1(1), & k = 5,8,11,\cdots \end{cases} \quad \textbf{(3.2a)}$$

General solution has been obtained.

Factoring sigma outside, we get

$$A(k) = \begin{cases} \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)} a_1(1), \quad k = 2 \\ \sum_{i=0}^{\frac{k-3}{3}} \left[c_2(3i+1) \frac{a_1(0)}{c_3(0)} \prod_{m=1}^{i} \frac{c_3(3m-2)}{c_3(3m)} - \frac{1}{b_2(0)} c_2(3i) \prod_{m=1}^{i} \frac{c_3(3m-3)}{c_3(3m-1)} \right] \\ \qquad + a_0(0), \quad k = 3,6,9,\cdots \\ \sum_{i=0}^{\frac{k-1}{3}} \left[\frac{c_2(3i-1)}{c_3(3i-1)} a_1(1) \prod_{m=1}^{i} \frac{c_3(3m-1)}{c_3(3m-2)} - \frac{a_1(0)}{c_3(0)} c_2(3i-2) \prod_{m=1}^{i-1} \frac{c_3(3m-2)}{c_3(3m)} \right] \\ \qquad + a_0(1), \quad k = 4,7,10,\cdots \\ \sum_{i=1}^{\frac{k-2}{3}} \left[c_2(3i) \frac{1}{b_2(0)} \prod_{m=1}^{i} \frac{c_3(3m-3)}{c_3(3m-1)} - c_2(3i-1) \frac{a_1(1)}{c_3(1)} \prod_{m=1}^{i-1} \frac{c_3(3m-1)}{c_3(3i+1)} \right] \\ \qquad + \frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)} a_1(1), \quad k = 5,8,11,\cdots \end{cases} \quad \textbf{(3.2b)}$$

Now, considering the equations found, the steps of how to realize the third order system C as the sequential connection of the first and second order Subsystems A and B which are commutative with each other, can be listed as the follows;

Step 1: Select 7 coefficients properly by considering Step 7 for $k = 0,1,\ldots,6$;

$$a_0(0)\,, a_0(1), a_1(0) \neq 0, a_1(1) \neq 0\,,$$
$$b_0(0), b_1(0), b_2(0) \neq 0.$$

Step 2: Compute $a_1(k)$ using Equation (3.1b)

$$a_1(k) = \begin{cases} \dfrac{c_3(0)}{b_2(0)}, & k = 2 \\ a_1(0)\displaystyle\prod_{i=1}^{\frac{k}{3}} \dfrac{c_3(3i-2)}{c_3(3i-3)}, & k = 3,6,9,\cdots \\ a_1(1)\displaystyle\prod_{i=1}^{\frac{k-1}{3}} \dfrac{c_3(3i-1)}{c_3(3i-2)}, & k = 4,7,10,\cdots \\ \dfrac{c_3(0)}{b_2(0)}\displaystyle\prod_{i=1}^{\frac{k-2}{3}} \dfrac{c_3(3i)}{c_3(3i-1)}, & k = 5,8,11,\cdots \end{cases} \quad \textbf{(3.2c)}$$

Step 3: Compute $b_2(k)$ by inserting the found values of $a_1(k)$ into Eq. (3.1a)

$$b_2(k) = \begin{cases} \dfrac{c_3(0)}{a_1(0)}, & k = 1 \\ \dfrac{c_3(1)}{a_1(1)}, & k = 2 \\ b_2(0)\displaystyle\prod_{i=1}^{\frac{k}{3}} \dfrac{c_3(3i-1)}{c_3(3i-3)}, & k = 3,6,9,\cdots \\ \dfrac{c_3(0)}{a_1(0)}\displaystyle\prod_{i=1}^{\frac{k-1}{3}} \dfrac{c_3(3i)}{c_3(3i-2)}, & k = 4,7,10,\cdots \\ \dfrac{c_3(1)}{a_1(1)}\displaystyle\prod_{i=1}^{\frac{k-2}{3}} \dfrac{c_3(3i+1)}{c_3(3i-1)}, & k = 5,8,11,\cdots \end{cases} \quad \textbf{(3.2d)}$$

Step 4: Compute $b_1(k)$ by using Eq. (3.1b) and (3.1d) in Eq. (3.1c)

$$
b_1(k) = \begin{cases}
\frac{c_2(0)}{a_1(0)} - \frac{b_2(0)}{a_1(0)} a_0(0), \quad k = 1 \\
\frac{c_2(1)}{a_1(1)} - \frac{c_3(0)a_0(1)}{a_1(0)a_1(1)}, \quad k = 2 \\
\prod_{i=1}^{\frac{k}{3}} \frac{c_3(3i-2)}{c_3(3i-3)} \left\{ a_1(0) \sum_{i=1}^{\frac{k-3}{3}} \left[\frac{c_2(3i-1)}{a_1(3i-1)a_1(3i)} - \frac{c_2(3i)}{a_1(3i)a_1(3i+1)} \right] + \frac{b_2(0)b_1(0)}{a_1(1)} \prod_{i=1}^{\frac{k}{3}} \frac{c_3(3i-2)}{c_3(3i-3)} \right\}, \quad k = 3,6,9,\cdots \\
\prod_{i=1}^{\frac{k-1}{3}} \frac{c_3(3i-1)}{c_3(3i-2)} \left\{ a_1(1) \sum_{i=1}^{\frac{k-1}{3}} \left[\frac{c_2(3i)}{a_1(3i)a_1(3i+1)} - \frac{c_2(3i-2)}{a_1(3i-2)a_1(3i-1)} \right] + \frac{c_2(0)}{a_1(0)} - \frac{b_2(0)a_0(0)}{a_1(0)} \right\}, \quad k = 4,7,10,\cdots \\
\prod_{i=1}^{\frac{k-2}{3}} \frac{c_3(3i)}{c_3(3i-1)} \left\{ \frac{c_2(1)}{a_1(1)} + \frac{c_3(0)}{b_2(0)} \sum_{i=1}^{\frac{k-2}{3}} \left[\frac{c_2(3i+1)}{a_1(3i+1)a_1(3i+2)} - \frac{c_2(3i-1)}{a_1(3i-1)a_1(3i)} \right] - \frac{a_0(1)c_3(0)}{a_1(0)a_1(1)} \right\}, \quad k = 5,8,11,\cdots
\end{cases} \quad \textbf{(3.2e)}
$$

Step 5: Compute $a_0(k)$ from Eq. (3.1e) or Eq. (3.2a)

$$
a_0(k) = \begin{cases}
-\frac{b_1(0)}{b_2(0)} a_1(1) + A(0), \quad k = 2 \\
\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(0), \quad k = 3,6,9,\cdots \\
\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(1), \quad k = 4,7,10,\cdots \\
\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} - \frac{b_1(0)}{b_2(0)} a_1(1), \quad k = 5,8,11,\cdots
\end{cases} \quad \textbf{(3.3a)}
$$

Step 6: Compute $b_0(k)$ by inserting Eq. (3.1e) into Eq. (3.1d)

$$
b_0(k) = \begin{cases}
\dfrac{b_1(0)}{a_1(0)}[a_0(k)-a_0(0)]+b_0(0), \quad k=1 \\[2ex]
\left[\dfrac{a_1(1)}{b_2(0)}\sum_{i=0}^{\frac{k-3}{3}}\left(\begin{array}{c}\dfrac{c_2(3i+1)}{a_1(3i+1)a_1(3i+2)} \\ -\dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)}\end{array}\right)\right]\left\{\left[\sum_{i=0}^{\frac{k-3}{3}}\left(\begin{array}{c}\dfrac{c_2(3i+1)}{a_1(3i+1)a_1(3i+2)} \\ -\dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)}\end{array}\right)\right.\right. \\
\left.\left. +\dfrac{a_0(0)}{a_1(0)}\dfrac{b_2(0)}{a_1(1)}-\dfrac{a_0(1)}{a_1(0)}\dfrac{b_2(0)}{a_1(1)}\right]+\dfrac{b_1(0)}{a_1(0)}\right\}+b_0(0), \qquad k=3,6,9,\cdots \\[2ex]
\left[\dfrac{a_1(1)}{b_2(0)}\sum_{i=1}^{\frac{k-1}{3}}\left(\dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)}-\dfrac{c_2(3i-2)}{a_1(3i-2)a_1(3i-1)}\right)+\dfrac{a_0(1)}{a_1(0)}\right. \\
\left.-\dfrac{a_0(0)}{a_1(0)}\right]\left\{\left[\sum_{i=1}^{\frac{k-1}{3}}\left(\begin{array}{c}\dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)} \\ -\dfrac{c_2(3i-2)}{a_1(3i-2)a_1(3i-1)}\end{array}\right)\right]+\dfrac{b_1(0)}{a_1(0)}\right\}+b_0(0), \quad k=4,7,10,\cdots \\[2ex]
\left[\dfrac{a_1(1)}{b_2(0)}\sum_{i=1}^{\frac{k-2}{3}}\left(\begin{array}{c}\dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)} \\ -\dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)}\end{array}\right)-\dfrac{a_1(1)}{a_1(0)}\dfrac{b_1(0)}{b_2(0)}\right. \\
\left.+\dfrac{1}{a_1(0)}\dfrac{c_2(0)}{b_2(0)}-\dfrac{a_0(0)}{a_1(0)}\right]\left\{\left[\sum_{i=1}^{\frac{k-2}{3}}\left(\begin{array}{c}\dfrac{c_2(3i)}{a_1(3i)a_1(3i+1)} \\ -\dfrac{c_2(3i-1)}{a_1(3i-1)a_1(3i)}\end{array}\right)\right.\right. \\
\left.\left.\begin{array}{c}-\dfrac{b_1(0)}{a_1(1)}\dfrac{1}{a_1(0)}+\dfrac{c_2(0)}{a_1(0)} \\ -\dfrac{b_2(0)}{a_1(1)}\dfrac{a_0(1)}{a_1(0)}\end{array}\right]+\dfrac{b_1(0)}{a_1(0)}\right\}+b_0(0), \quad k=2,5,8,\cdots
\end{cases} \tag{3.3b}
$$

Or, by putting values $a_0(k)$ in Eq. (3.2a) into Eq. (3.1d)

$$
\begin{aligned}
&b_0(k)\\
&=\begin{cases}
\dfrac{1}{a_1(0)}\left[-\dfrac{b_1(0)}{b_2(0)}a_1(1)+A(0)-a_0(0)\right]\left\{\left[-b_1(0)+\dfrac{b_2(0)}{a_1(1)}A(0)\right.\right. \\
\qquad \left.\left.-\dfrac{b_2(0)}{a_1(1)}a_0(1)\right]+\dfrac{b_1(0)}{a_1(0)}\right\}+b_0(0), \quad k=2 \\
\dfrac{1}{a_1(0)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}-\dfrac{A(k-3)}{A(k-2)A(k-3)}\right]\left\{\dfrac{b_2(0)}{a_1(1)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}\right.\right. \\
\qquad \left.\left.-\dfrac{A(k-3)}{A(k-2)A(k-3)}+a_0(0)-a_0(1)\right]+\dfrac{b_1(0)}{a_1(0)}\right\}+b_0(0), \quad k=3,6,9,\cdots \\
\dfrac{1}{a_1(0)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}-\dfrac{A(k-3)}{A(k-2)A(k-3)}+a_0(1)\right. \\
\qquad \left. a_0(0)\right]\left\{\dfrac{b_2(0)}{a_1(1)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}-\dfrac{A(k-3)}{A(k-2)A(k-3)}\dfrac{b_1(0)}{a_1(0)}\right\}\right. \\
\qquad +b_0(0), \quad k=4,7,10,\cdots \\
\dfrac{1}{a_1(0)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}-\dfrac{A(k-3)}{A(k-2)A(k-3)}-\dfrac{b_1(0)}{b_2(0)}a_1(1)-a_0(0)\right] \\
\dfrac{b_2(0)}{a_1(1)}\left[\dfrac{A(k-2)}{A(k-1)A(k-2)}-\dfrac{A(k-3)}{A(k-2)A(k-3)}-\dfrac{b_1(0)}{b_2(0)}a_1(1)-a_0(1)\right] \\
\qquad +\dfrac{b_1(0)}{a_1(0)}+b_0(0), \quad k=5,8,11,\cdots
\end{cases}
\end{aligned}
$$

$$
b_0(k) = \begin{cases}
\frac{1}{a_1(0)}\left[-\frac{b_1(0)}{b_2(0)}a_1(1) + A(0) - a_0(0)\right]\left\{\left[-b_1(0) + \frac{b_2(0)}{a_1(1)}A(0) - \frac{b_2(0)}{a_1(1)}a_0(1)\right] + \frac{b_1(0)}{a_1(0)}\right\} + b_0(0), \quad k = 2 \\
\frac{1}{a_1(0)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)}\right]\left\{\frac{b_2(0)}{a_1(1)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(0) - a_0(1)\right] + \frac{b_1(0)}{a_1(0)}\right\} + b_0(0), \quad k = 3,6,9,\cdots \\
\frac{1}{a_1(0)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} + a_0(1) \; a_0(0)\right]\left\{\frac{b_2(0)}{a_1(1)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} \frac{b_1(0)}{a_1(0)}\right\} + b_0(0), \quad k = 4,7,10,\cdots \\
\frac{1}{a_1(0)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} - \frac{b_1(0)}{b_2(0)}a_1(1) - a_0(0)\right]\left\{\frac{b_2(0)}{a_1(1)}\left[\frac{A(k-2)}{A(k-1)A(k-2)} - \frac{A(k-3)}{A(k-2)A(k-3)} - \frac{b_1(0)}{b_2(0)}a_1(1) - a_0(1)\right] + \frac{b_1(0)}{a_1(0)}\right\} + b_0(0), \quad k = 5,8,11,\cdots
\end{cases} \tag{3.3c}
$$

Step 7: Check the condition in Eq. (2.8d) for $k = 0,1,\ldots 6$ when applying Step 1; This equation must be satisfied for all $k \geq 7$; if not satisfied, return to Step 1 and try to rechoose proper initial values for the coefficients; the decomposition is not possible if this choice of proper initial values is not possible, then stop the trial. If it is satisfied proceed to the next step.

Step 8: the choice of the initial condition.

8a. If the initial conditions are zero; that is, if $y(0) = y(1) = y(2) = 0$,

$$y_{A0} = y_{B0} = y_{B1} = 0.$$

8b. If the initial conditions are not zero; that is, $y(0) \neq 0$, then from Eqs. (2.9a,b,c), the following must be satisfied;

$$y(0) = y_{A0} = y_{B0} \tag{3.4a}$$

$$y(1) = y_{B1} = \frac{1 - a_0(0)}{a_1(0)}y(0) \tag{3.4b}$$

$$y(2) = \frac{a_1(0) - b_1(0) + b_1(0)a_0(0) - a_1(0)b_0(0)}{b_2(0)a_1(0)} y(0)$$
$$= \frac{1 - a_0(0) - a_0(1) + a_0(1)a_0(0)}{a_1(1)a_1(0)} y(0), \quad \textbf{(3.4c)}$$

The last equation is reexpression of Eq. (2.10) for $y(0) \neq 0$.

This means that if the initial conditions are nonzero, the random selections in step 1 must satisfy the conditions in step (8b). The results found so far can be summarized in Theorem 3.1 and Theorem 3.2.

Theorem 3.1: The necessary and sufficient conditions for a discrete time linear time-varying system C defined by the difference equation in (2.7) to be decomposed as the cascade connection of first order and second order discrete-time linear time-varying subsystems A and B (as defined by the difference Eqs. (2.1a) and (2.2a), respectively) are;

i. Let $a_0(0),\ a_0(1), a_1(0) \neq 0\ , a_1(1);\ b_0(0), b_1(0), b_2(0) \neq 0$ be properly chosen constants so that the third condition below is satified for $k = 0,1, \dots, 6$.

ii. The coefficients of A and B as defined in Eqs. (2.1a) and (2.2a) be selected by the formulas

$a_1(k)$: Eq. (3.1b)

$a_0(k)$: Eq. (3.1e) or Eq. (3.2a)

$b_2(k)$: Eq. (3.3)

$b_1(k)$: Eq. (3.4)

$b_0(k)$: Eq. (3.6a) or Eq. (3.6b)

iii. With the computed values Eq. (2.8d), that is $c_0(k) = a_0(k)b_0(k)$, must also be satisfied for all $k \geq 7$.

Theorem 3.2: A third-order linear time-varying discrete-time system C defined by Eq. (2.7a) with nonzero conditions defined in Eq. (2.7b) can be decomposed as the cascade connection of commutative first-order and second order linear discrete time time-varying Subsystems A and B represented as defined in Eqs. (2.1) and (2.2) if and only if the necessary and sufficient conditions of Theorem 3.1 are satisfied; moreover, the initial conditions of A, B, C and the properly chosen initial values of the coefficients mentioned in item i. of Theorem 1 should satisfy the conditions set by Eqs. (7ab, c); specially the following equation which results from Eq. (7c) should hold if $y(0) \neq 0$:

$$\frac{a_1(0)[1-b_0(0)]-b_1(0)[1-a_0(0)]}{b_2(0)}=\frac{[1-a_0(0)][1-a_0(1)]}{a_1(1)} \qquad \textbf{(3.4d)}$$

Before finishing this section, let us create simpler alternative formulas for $a_0(k)$ and b_0(k). Since $a_1(k)$ is known from Eq. (3.1b), we can obtain $a_0(k)$ by iteratively solving Eq. (3.2a). Substituting 0, 1, 2,… for k in Eq. (3.2a) and using the value $a_0(k)$ found before each time

$$a_0(2)=\frac{c_2(0)}{b_2(0)}-\frac{b_1(0)}{b_2(0)}a_1(1),$$

$$a_0(3)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(1)}{a_1(2)}\right]-\frac{c_2(0)}{b_2(0)}+a_0(0),$$

$$a_0(4)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(2)}{a_1(2)a_1(3)}-\frac{c_2(1)}{a_1(2)}\right]+a_0(1),$$

$$a_0(5)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(3)}{a_1(3)a_1(4)}-\frac{c_2(2)}{a_1(2)a_1(3)}\right]+\frac{c_2(0)}{b_2(0)}-\frac{b_1(0)}{b_2(0)}a_1(1),$$

$$a_0(6)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(4)}{a_1(4)a_1(5)}-\frac{c_2(3)}{a_1(3)a_1(4)}+\frac{c_2(1)}{a_1(2)}\right]-\frac{c_2(0)}{b_2(0)}+a_0(0),$$

$$a_0(7)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(5)}{a_1(5)a_1(6)}-\frac{c_2(4)}{a_1(4)a_1(5)}+\frac{c_2(2)}{a_1(2)a_1(3)}-\frac{c_2(1)}{a_1(2)}\right]+a_0(1),$$

$$a_0(8)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(6)}{a_1(6)a_1(7)}-\frac{c_2(5)}{a_1(5)a_1(6)}+\frac{c_2(3)}{a_1(3)a_1(4)}-\frac{c_2(2)}{a_1(2)a_1(3)}\right]+\frac{c_2(0)}{b_2(0)},$$

$$a_0(9)=\frac{a_1(0)a_1(1)}{b_2(0)}\left[\frac{c_2(7)}{a_1(7)a_1(8)}-\frac{c_2(6)}{a_1(6)a_1(7)}+\frac{c_2(4)}{a_1(4)a_1(5)}-\frac{c_2(3)}{a_1(3)a_1(4)}+\frac{c_2(1)}{a_1(2)}\right]$$

are obtained. The overall result can be written as in Eq. (3.5a);

$$a_0(k)=\begin{cases} -\dfrac{c_2(0)}{b_2(0)}+a_0(0), & k=2 \\ \mathcal{A}(k)-\dfrac{c_2(0)}{b_2(0)}+a_0(0), & k=3,6,9,\cdots \\ \mathcal{A}(k)+a_0(1), & k=4,7,10,\cdots \\ \mathcal{A}(k)+\dfrac{c_2(0)}{b_2(0)}-\dfrac{b_1(0)}{b_2(0)}a_1(1), & k=5,8,11,\cdots \end{cases} \qquad \textbf{(3.5a)}$$

$$\mathcal{A}(k) = \begin{cases} \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1), \quad k = 2 \\ \dfrac{a_1(0)}{b_2(0)} a_1(1) \displaystyle\sum_{i=1}^{k-2} (-1)^{i+1} \dfrac{c_2(i)}{a_1(i)a_1(i+1)}, \quad k = 3,4,5,\cdots \end{cases} \tag{3.5b}$$

Finally, using equation (3.1d)

,

$$b_0(k) = \begin{cases} \dfrac{b_2(0)}{a_1(0)a_1(1)} \left[\mathcal{A}(k) - \dfrac{c_2(0)}{b_2(0)}\right] \left[\mathcal{A}(k) - \dfrac{c_2(0)}{b_2(0)} + a_0(0) - a_0(1)\right] \\ \quad + \dfrac{b_1(0)}{a_1(0)} \left[\mathcal{A}(k) - \dfrac{c_2(0)}{b_2(0)}\right] + b_0(0), \quad k = 3,6,9,\cdots \\ \dfrac{b_2(0)}{a_1(0)a_1(1)} [\mathcal{A}(k) + a_0(1) - a_0(0)][\mathcal{A}(k)] \\ \quad + \dfrac{b_1(0)}{a_1(0)} [\mathcal{A}(k) + a_0(1) - a_0(0)] + b_0(0), \quad k = 4,7,10,\cdots \\ \dfrac{b_2(0)}{a_1(0)a_1(1)} \left[\mathcal{A}(k) + \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1) - a_0(0)\right] [\mathcal{A}(k) \\ \quad + \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1) - a_0(1)\Big] + \dfrac{b_1(0)}{a_1(0)} [\mathcal{A}(k) \\ \quad + \dfrac{c_2(0)}{b_2(0)} - \dfrac{b_1(0)}{b_2(0)} a_1(1) - a_0(0)\Big] + b_0(0), \quad k = 5,8,11,\cdots \end{cases} \tag{3.5c}$$

4. EXAMPLES, DISCUSSIONS AND RESULTS

4.1. Example 1

Let the third order time-varying discretete-time linear system C be defined as;

$$(k+1)^2 y(k+3) + (k+1)y(k+2) + 5ky(k+1) + y(k) = x(k), k = 0,1 \quad \textbf{(4.1a)}$$

$$y(0) = y(1) = y(2) = 0, \quad \textbf{(4.1b)}$$

where

$$c_3 = (k+1)^2, c_2 = (k+1), c_1 = 5k, c_0 = 1, \quad \textbf{(4.1c)}$$

Choose the inital values according to Item i. of Theorem 1.

$$a_0(0) = 0, a_0(1) = 1, a_1(0) = 1, a_1(1) = 1, b_0(0) = 0, b_1(0) = b_2(0) = 1 \quad \textbf{(4.1d)}$$

Using Eq. (3.1b)

$$a_1(k) = \begin{pmatrix} 1, \quad k = 2 \\ \dfrac{2\sqrt{3}\pi\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)}{3\left[\Gamma\left(\frac{2}{3}\right)\right]^2 \Gamma\left(\frac{k}{3}+\frac{1}{3}\right)}, \quad k = 3,6,9,12,\cdots \\ \dfrac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)}{\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)}, \quad k = 4,7,10,\cdots \\ \dfrac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)}{2\pi\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)}, \quad k = 5,8,11,\cdots \end{pmatrix}^2 \quad \textbf{(4.2a)}$$

From Eq. (3.8ac), the following equation is found for $a_0(k)$:

$$a_0(k) = \begin{cases} \mathcal{A}(k), & k = 2,5,8,\cdots \\ \mathcal{A}(k) - 1, & k = 3,6,9,\cdots \\ \mathcal{A}(k) + 1, & k = 4,7,1,\cdots \end{cases} \quad \textbf{(4.2b)}$$

.

From equation (3.8b) the following equation is obtained for $\mathcal{A}(k)$:

$$\mathcal{A}(k) = \begin{cases} 0, & k = 2 \\ \displaystyle\sum_{i=1}^{k-2} (-1)^{i+1} \frac{i+1}{a_1(i)a_1(i+1)}, & k = 3,4,5,\cdots \end{cases} \tag{4.2c}$$

From Eq. (3.1a), the following equation is obtained for $b_2(k)$:

$$b_2(k) = \left(\begin{array}{c} 1, \quad k = 1 \\ 2, \quad k = 2 \\ \dfrac{2\sqrt{3}\pi\Gamma\left(\frac{k}{3}+1\right)}{3\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}, \quad k = 3,6,9,12,\cdots \\ \dfrac{3\sqrt{3}\left[\Gamma\left(\frac{2}{3}\right)\right]^2\Gamma\left(\frac{k}{3}+1\right)}{2\pi\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)}, \quad k = 4,7,10,\cdots \\ \dfrac{3\left(\frac{k}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)}, \quad k = 5,8,11,\cdots \end{array} \right)^2 \tag{4.3a}$$

Using Eq. (3.2e):

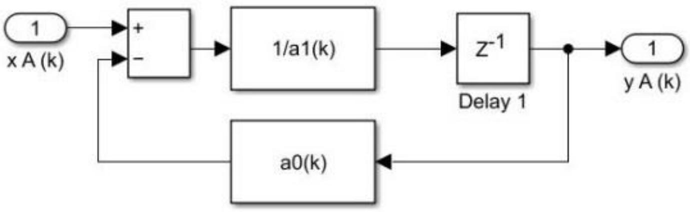
$$b_1(k) = \begin{cases} 1, \quad k = 2 \\ \dfrac{2\sqrt{3}\pi k\Gamma(k+1)}{9\Gamma\left(k+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} - \dfrac{\Gamma\left(k+\frac{1}{3}\right)\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)\left[\frac{k}{6}(k-3)+1\right]}{\Gamma(k+1)\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \\ \quad - \dfrac{2\sqrt{3\pi\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)}}{3\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)\left[\Gamma\left(\frac{2}{3}\right)\right]^2}, \quad k = 3,6,9,\cdots \\ \dfrac{\sqrt{3}\Gamma\left(k+\frac{1}{3}\right)\left[\Gamma\left(\frac{2}{3}\right)\right]^2\left[\frac{1}{6}(k-1)(k+2)+1\right]}{2\pi\Gamma\left(k+\frac{2}{3}\right)} \\ \quad - \dfrac{\sqrt{3}\pi\Gamma(k+1)\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)(k-1)(k+2)}{27\Gamma\left(k+\frac{4}{3}\right)\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)} \\ \quad + \dfrac{[\Gamma(k+1)]^2\left[\Gamma\left(\frac{2}{3}\right)\right]^2\left(\frac{k}{3}-\frac{1}{3}\right)}{\left[\Gamma\left(k+\frac{2}{3}\right)\right]^2} \\ \quad - \dfrac{2\sqrt{3}\pi\Gamma(k)\left[\frac{1}{6}(k+2)(k-3)+1\right]}{9\Gamma\left(k+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}, \quad k = 4,7,10,\cdots \\ \dfrac{\Gamma\left(\frac{k}{3}+1\right)(2k+11)\,(k+1))}{10\Gamma\left(\frac{k}{3}+\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)} \\ \quad - \dfrac{27\left[\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)\right]^2\left[\Gamma\left(\frac{2}{3}\right)\right]^2 k\Gamma\left(\frac{k}{3}+\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)}{2\Gamma\left(\frac{k}{3}+1\right)} - \dfrac{3}{2}, \quad k = 2,5,8,\cdots \end{cases} \quad \textbf{(4.3b)}$$

From Eq. (3.3a), the following equation is obtained for $b_0(k)$:

$$b_0(k) = \begin{cases} 1, \quad k = 2 \\ \left[\frac{k}{3} + \frac{Psi\left(\frac{k}{3}+\frac{1}{3}\right)}{3} + \frac{1}{2}ln3 + \frac{\sqrt{3}\pi}{18} - \frac{2\sqrt{\pi}k\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)(k+1)-1)}{8\pi\Gamma\left(k+\frac{1}{3}\right)}\right]^2, \quad k = 3,6,9,\ldots \\ \left[\frac{\Gamma\left(\frac{k}{3}+\frac{5}{3}\right)\Gamma\left(\frac{2}{3}\right)(4k-1)(k+1)}{28\Gamma\left(\frac{k}{3}+\frac{4}{3}\right)} - \frac{\sqrt{3}\pi\Gamma\left(\frac{k}{3}+\frac{1}{3}\right)(k+5)+\frac{33}{28}}{18\Gamma\left(\frac{k}{3}+1\right)\left(\Gamma\left(\frac{2}{3}\right)\right)^2}\right]^2, \quad k = 4,7,10,\ldots \\ \left[\frac{\sqrt{3}\Gamma\left(\frac{k}{3}+\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)(k+1)(k+2)}{8\Gamma\left(\frac{k}{3}+\frac{4}{3}\right)} - \frac{\Gamma\left(\frac{k}{3}+\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)(k+5)+\frac{2}{3}}{2\Gamma\left(\frac{k}{3}+1\right)}\right]^2, \quad k = 5,8,11,\ldots \end{cases} \quad \textbf{(4.3c)}$$

MATLAB Simulink diagrams for A, B and C are shown in Fig. 4.1a, b, c, respectively. Simulation results with the coefficients expressed by Eqs. (3) are shown in Fig. 4.2. A 10-amplitude sine signal is used at the frequency of π/4 rad/s as an input. As the figure shows, subsystems A and B formed a commutative pair, that is, AB and BA give the same output signal but this signal is not equal to the output signal of C. As a result, A and B failed to decompose C into its commutative subsystems. This is because the conditions of Theorem 3.1 has not been met. In fact, Eq. (2.8d) which is Condition iii. of Theorem 3.1 is not generally provided for k values, as can be seen from Table 4.1.

(a)



(b)

(c)

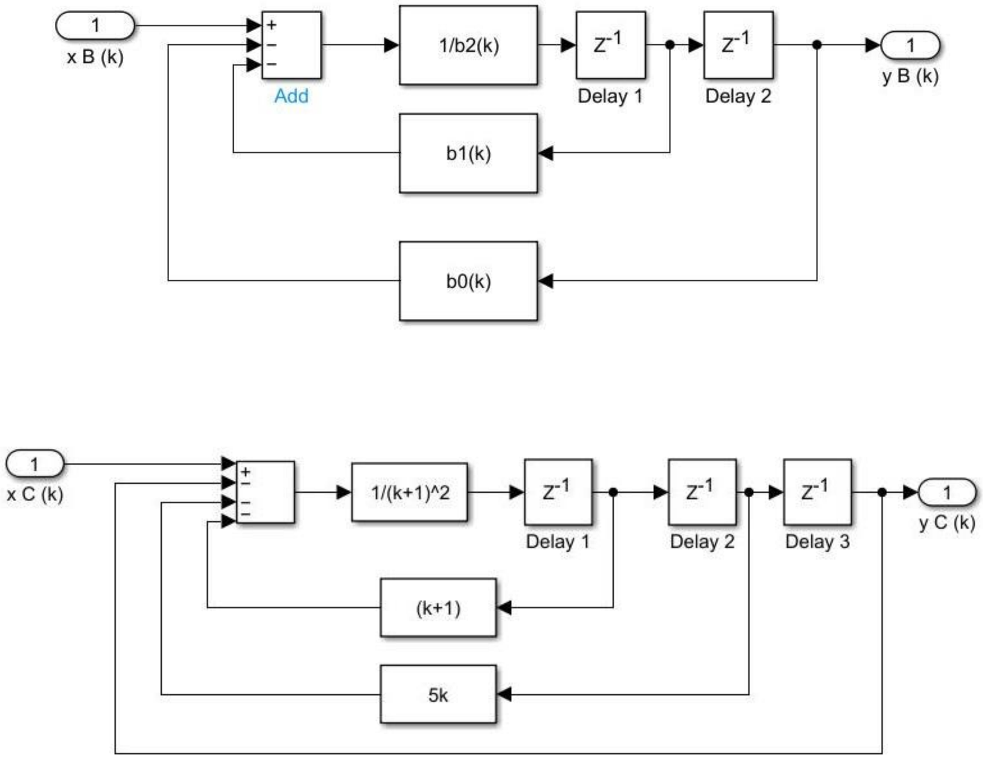


Figure 4.1. Block schemas of (a) Subsystem A, (b) Subsystem B, (c) System C

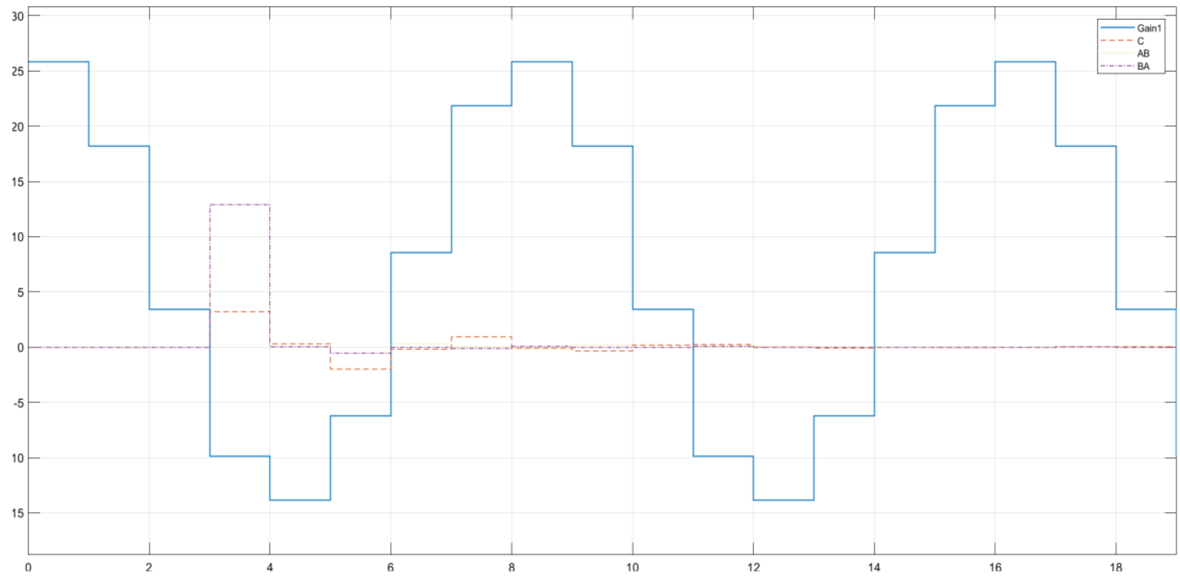


Figure 4.2. Response functions of Systems AB, BA, C; incorrect decomposition

Table 4.1. Numerical values of coefficients for Subsystems A and B of Ex. 1

k	$a_1(k)$	$a_0(k)$	$b_2(k)$	$b_1(k)$	$b_0(k)$	$b_0(k) * a_0(k)$	$c_0(k)$
0	1	0	1	0	0	0	1
1	1	0	1	0	0	0	1

2	1	0	4	0.694	0	0	1
3	4	0.6944	0	8.556	0.4823	0.3349	1
4	2.25	1.444	258.8	5.465	2.086	3.014	1
5	1.778	0.9844	11.11	1.108	0.9691	0.9541	1
6	6.25	-0.3612	20.25	14.36	0.1305	-0.04714	1
7	3.24	2.659	3914	12.52	7.071	18.8	1
8	2.42	1.204	316	4.128	1.449	1.748	1
9	8.163	0.5023	33.47	17.2	0.2523	0.1267	1
10	4.101	1.604	$3.092\text{x}10^5$	10.72	2.574	4.129	1

4.2. Example 2

In the above example, select System C to provide the third condition of Theorem 3.1. For this purpose, the $c_0(k)$ values in Chapter 2 are equal to the $a_0(k) * b_0(k)$ values in the table for $k \in [0,10]$. Therefore, at least the conditions of the theorem are provided upto $k = 12$. The simualtion results found without making any further changes in the previous example are given in Figure 4.3. As the way it is seen, the separation process was successful upto $k = 12$, and indeed Systems AB, BA, C give the same output signal. However, for $k \geq 12$, C is no longer equivalent to AB and BA consecutive connections and they give different results.

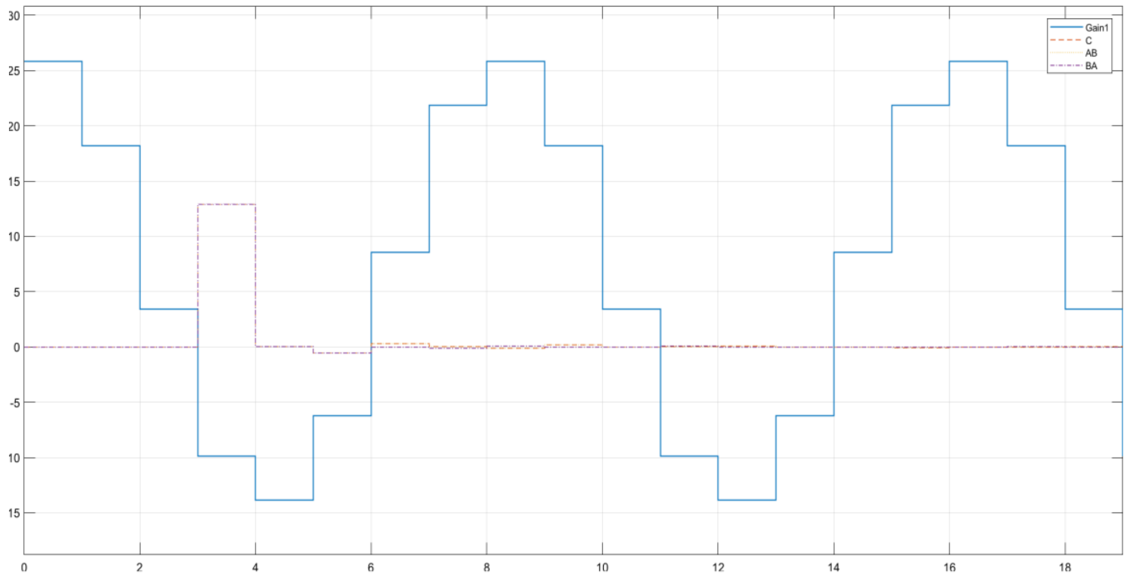


Figure 4.1. Simulation results for right decomposition of System C to Systems A and B

4.3. Example 3

Let the third-order linear discrete systems is chosen with coefficients

$$c_3(k) = (k^3 + 2k^2 - k + 3)(k + 2),$$

$$c_2(k) = (-k^2 - 3k + 6)(k + 2) + (k + 1)(k - 4),$$

$$c_1(k) = (k + 1)(2 - cos(0.1(k + 1)\pi) + (k - 4)(-k^2 - 3k + 6),$$

$$c_0(k) = (k - 4)(2 - cos(0.1(k + 1)\pi).$$

And, let the coefficients of A and B be as the following

$$a_1(k) = (k + 2), a_0(k) = (k - 4), b_2(k) = (k^3 + 2k^2 - k + 3), b_1(k) = (-k^2 - 3k + 6),$$

$$b_0(k) = (2 - cos(0.1(k + 1)\pi).$$

In this case, $a_1(0) = 2$, $a_0(0) = -4$, $b_2(0) = 3, b_1(0) = 6$, and $b_0(k) = (2 - cos(0.1(k + 1)\pi)$.

With initional condition

$$y(0) = y_{A0} = y_{B0} = 1.5,\ y(1) = 1.5, y(2) = 1.5.$$

Simulations with the input function used in Examples 4.1 and 4.2 confirm that Subsystems A and B are commutative decomposition of System C. Indeed, in Fig. 4, the output functions of AB, BA and C systems are seen to be equal to each other.

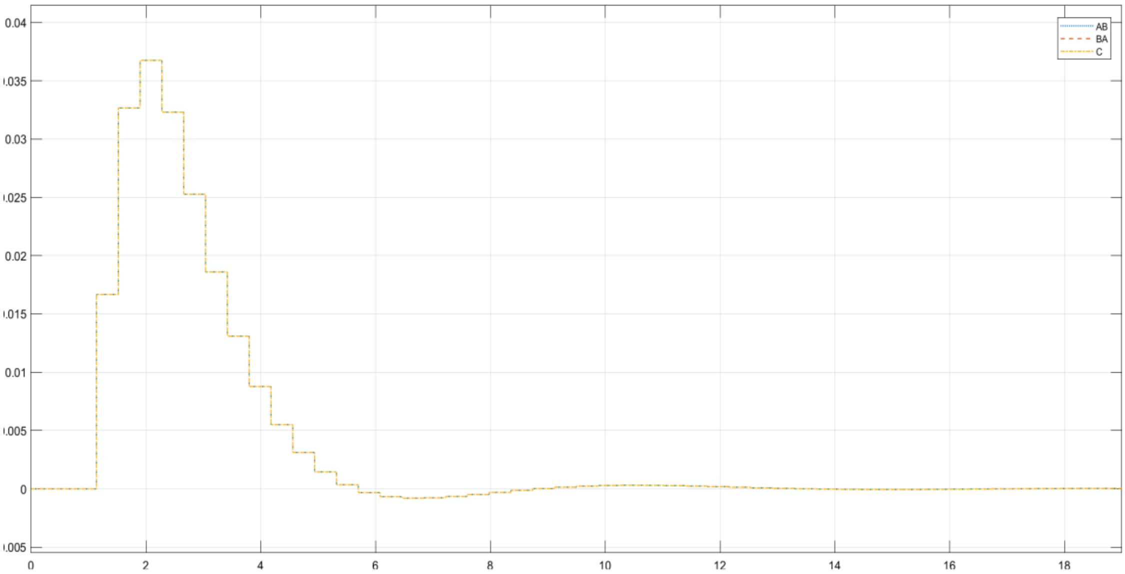


Figure 4.2. Response functions of *AB*, *BA* and *C* in Ex. 3

5. CONCLUSIONS AND RECOMMENDATIONS

In this study, we looked at how to decompose any third-order discrete-time linear time-varying system into its first- and second-order commutative pairs. For both zero and non-zero beginning conditions, explicit decomposition formulas are derived. Matlab simulations are used to verify the findings. The work is unique and is being published for the first time. It is critical from the standpoints of synthesis and/or design. Because many design techniques in engineering systems rely on tearing and reconstruction. This is the process of putting together simple components to create a finished product. Furthermore, it is demonstrated that regarding sensitivity to initial conditions, some combinations may be better than others. The results of this work can be extended for the decomposition of fourth-order discrete-time linear time-varying systems into lower order commutative pairs; as two second-order commutative subsystems or one first-order and one third-order commutative subsystems.

REFERENCES

Ibrahim, S. (2019). Commutativity of High-her Order Linear Time-varying Differential Systems. *PhD Thesis,* Ondokuz Mayis University, Samsun.

Koksal, M .ve Koksal, M. E. (2011). Commutativity of linear time-varying differential systems with non-zero initial conditions: A review and some new extensions. *Mathematical Problems in Engineering*, 1-25.

Koksal, M. (1982). Commutativity of second order time-varying systems. *International Journal of Control*, 36: 3, 541-544.

Koksal, M. (1988). Effects of nonzero initial conditions on the commutativity of linear time-varying systems, *International Conference on Modeling and Simulation,* vol. 1A, 49-55, June 29- July 1, Istanbul, Turkey.

Koksal, M. and Koksal, M. E. (2013). Ardışık Bağlı Zamanla Değişen Doğrusal Ayrık-zaman Sistemlerinin Sıra-değişim Özelliği, *Turkish National Meeting on Automatics Control,* 1128-1131, Sep 26-28, Malatya, Turkey.

Koksal, M. and Koksal, M. E. (2015). Commutativity of cascade connected discrete time linear varying systems. *Transactions of the Institute of Measurement and Control*, 37, 615-622.

Koksal, M. E. (2019a). Commutativity of systems with their feedback conjugates. *Transactions of the Institute of Measurement and Control*, 41: 3, 696-700.

Koksal, M. E. (2019b). Explicit commutativity conditions for second order linear time varying systems with non-zero initial conditions. *Archives of Control Sciences,* 29: 3, 413-432.

Koksal, M. E. (2019c). Transitivity of commutativity for second-order linear time-varying analogue systems. *Circuits, Systems and Signal Processing*, 38: 3, 1385-1395.

Koksal, M. E. (2019d). Commutativity of first-order discrete-time linear time-varying systems. *mathematical methods in the applied sciences*, 10.1002 / mma.5310.

Koksal, M. ve Koksal, M. E. (2015). Commutativity of cascade connected discrete time linear time-varying systems. *Transactions of the Institute of Measurement and Control,* 1-8.

Li, Y., Liu, S. and Wang, Z. (2016). Fault detection for linear discrete time-varying systems with intermittent observations and quantization errors. *Asian Journal of Control,* 18:1, 377-389.

Marshall, E. (1977). Commutativity of time-varying systems. *Electronics Letters*. 18, 539-540.

Nguyen, H. N., Olaru, S., Gutman, P. O. and Hovd, M. (2015). Constrained control of uncertain, time-varying linear discrete-time systems subject to bounded disturbances. *IEEE Transactions on Automatic Control,* 60:3, 831-836.

Nguyen, H. N., Olaru, S., Gutman, P. O. and Hovd, M. (2015). Constrained control of uncertain, time-varying linear discrete-time systems subject to bounded disturbances. *IEEE Transactions on Automatic Control,* 60:3, 831-836.

Phat, V. N., Ratichagit, K. (2011). Stability and stabilization of switched linear discrete-time systems with interval time-varying delay. *Nonlinear Analysis: Hybrid Systems,* 5, 605-612.

Shen, B.,Ding, S. T. and Wang, Z. (2013). Finite-horizon H∞ fault estimation for uncertain linear discrete time-varying systems with known inputs. *IEEE Transactions on Automatic Control,* 60:2, 902-906.

Tabataeipour, S. M. (2013). Active fault detection and isolation of discrete-time linear time-varying systems: a set-membership approach. *International Journal of Control*, 46:11, 1917-1933.

Willigenburg, L.Gerard Van and Koning Willem, L. De. (2015). Temporal stabilizability and compensability of time-varying linear discrete-time systems with white stochastic parameters. *European Journal of C*ontrol 23, 36–47.

APPENDIX

CHOICE OF PROPER INITIAL VALUES

$a_0(0),\ a_0(1), a_1(0) \neq 0, a_1(1); b_0(0), b_1(0), b_2(0) \neq 0$ FOR THE COEFFICINTS

a_0, a_1, b_0, b_1, b_2

Eq. (2.8.d) yields for $k = 0$

$$a_0(0)b_0(0) = c_0(0). \tag{A.1}$$

Eq. (2.8.d) yields for $k = 1$

$$a_0(1)b_0(1) = c_0(1)$$

Find $b_0(1)$ using Eq. (3.1d) for $k = 1$, and insert in the above equation;

$$a_0(1)\left[\frac{b_1(0)}{a_1(0)}[a_0(1) - a_0(0)] + b_0(0),\right] = c_0(1) \tag{A.2}$$

Eq. (2.8.d) yields for $k = 2$

$$a_0(2)b_0(2) = c_0(2)$$

Find $a_0(2)$ using Eq. (3.2a) for $k = 2$, $b_0(2)$ using Eq. (3.1d) for $k = 2$, and insert them in the above equation;

$$\left[\frac{c_2(0)}{b_2(0)} - \frac{b_1(0)}{b_2(0)}a_1(1)\right]b_0(2) = c_0(2) \tag{A.3}$$

$$b_0(k) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(k) - a_0(0)][a_0(k) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(k) - a_0(0)] + b_0(0) \tag{3.1d}$$

$$a_0(k)b_0(k) = c_0(k). \qquad (2.8d)$$

Inserting in $b_0(k)$;

$$a_0(k)\left[\frac{b_2(0)}{a_1(0)a_1(1)}[a_0(k) - a_0(0)][a_0(k) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(k) - a_0(0)] + b_0(0)\right] = c_0(k). \qquad (2.8d)$$

$$\left[\frac{b_2(0)}{a_1(0)a_1(1)}\right][a_0(k)]^3 + \left[\frac{b_2(0)[-a_0(1) - a_0(0)]}{a_1(0)a_1(1)} - \frac{b_1(0)}{a_1(1)}\right][a_0(k)]^2 + \left[\frac{b_2(0)a_0(0)a_0(1)}{a_1(0)a_1(1)} - \frac{b_1(0)a_0(0)}{a_1(0)} + b_0(0)\right]a_0(k) = c_0(k)$$

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